Chapter 6: Introduction to Credibility

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Preview

Credibility is a method of estimation which balances the data of the past experience and the insurance company's expectation. It is commonly used for ratemaking and estimating claim frequencies, etc. In this chapter, we introduce the notion of limited fluctuation credibility, which can be categorized as full and partial credibility.

Key topics in this chapter:

1. Limited Fluctuation Credibility;
2. Full Credibility;
3. Partial Credibility.

1 Limited Fluctuation Credibility

Credibility is a method to estimate a quantity of interest (e.g., expected claim frequency, claim size, aggregate loss) by combining the (recent) past experience, and the company's or industry's expectations. Mathematically, let P be the ***credibility estimate*** of the quantity of interest, which is defined as

$$\boxed{P = ZY + (1-Z)M,} \tag{1}$$

where

- Y is the observation from data, e.g., the claim experience of a policyholder;
- M is the prior/general estimate, which is deterministic and determined based on the company's expected experience.
- $Z \in [0,1]$ is the ***credibility factor***.

To obtain the estimate P, the crucial step is to determine the factor Z which balances the company's expected and actual experience. When $Z = 0$, the premium is determined completely based on the expected experience; and when $Z = 1$, the prior knowledge is ignored and the estimate is determined solely based on the past experience.

In Equation (1), M is a deterministic factor, and Y is random in nature. Conceivably, as Z increases, the fluctuation of P increases, since more weight is put on the random component. To reduce the fluctuation of P, we may as well set $Z \to 1$, but this fails to reflect the actual experience of losses. ***Limited fluctuation credibility***, (a.k.a. ***classical credibility***), is a method to balance the two competing factors by limiting the fluctuations in the estimate. In this chapter, we will discuss two schemes:

- ***Full Credibility:*** determine the conditions on setting $Z = 1$, i.e., $P = Y$;
- ***Partial Credibility:*** determine the credibility factor Z when setting $Z = 1$ is not appropriate.

2 Full Credibility

We assign full credibility to the experience Y if it is "stable". More precisely, given a *range parameter* $k > 0$, and a *confidence level* $p \in (0, 1)$, full credibility is assigned to the data if the probability that Y lies in $[\mu_Y - k\mu_Y, \mu_Y + k\mu_Y]$ is at least p:

Definition 2.1 Full credibility is assigned to the experience Y with confidence level $p \in (0, 1)$ and range parameter $k > 0$ if

$$\boxed{\mathbb{P}\left(|Y - \mu_Y| \le k\mu_Y\right) \ge p.} \tag{2}$$

Let $\sigma_Y^2 = \mathrm{Var}[Y]$. We can rewrite the full credibility standard (2) as

$$\mathbb{P}\left(\left|\frac{Y - \mu_Y}{\sigma_Y}\right| \le \frac{k\mu_Y}{\sigma_Y}\right) \ge p.$$

Let y_p be the smallest value that satisfies

$$\mathbb{P}\left(\left|\frac{Y - \mu_Y}{\sigma_Y}\right| \le y_p\right) \ge p.$$

In other words, y_p is the $100p$-percentile of the random variable $|(Y - \mu_Y)/\sigma_Y|$. Then, full credibility is assigned if

$$\boxed{y_p \le \frac{k\mu_Y}{\sigma_Y} \iff \mathrm{CV}_Y \le \frac{k}{y_p},} \tag{3}$$

where $\mathrm{CV}_Y := \sigma_Y/\mu_Y$ is the coefficient of variation of Y.

Remark 2.1.

- In exam questions, we can approximate the distribution $(Y - \mu_Y)/\sigma_Y$ by a standard normal distribution, $\mathcal{N}(0, 1)$. In that case, we can approximate y_p by $z_{(p+1)/2}$, where

$z_{(p+1)/2} = \Phi^{-1}((p+1)/2)$. Hence, the full credibility standard reads as

$$\boxed{\mathrm{CV}_Y \leq \frac{k}{z_{(1+p)/2}}.} \tag{4}$$

- We can rearrange Equation (2) (resp. (4)) as

$$\boxed{\mu_Y \geq \frac{y_p \sigma_Y}{k} \quad \left(\text{resp. } \mu_Y \geq \frac{z_{(1+p)/2}\sigma_Y}{k}\right).}$$

  Intutively, the full credibility standard holds if μ_Y is sufficiently large.

- The parameter μ_Y is usually unknown: it is the parameter we want to estimate in most cases. To establish the full credibility standard, we will use the statistics from the sample (e.g., the sample mean) to estimate μ_Y in (2) or (4). This estimate will depend on the number of observations or ***exposures*** in the data. Hence, the full credibility standard holds if the sample size is sufficiently large, which limits the fluctuation of Y.

2.1 Full Credibility for Claim Frequency

Let N be the random variable of number of claims. We are going to use the observed number of claims to update our estimate of the true expected number of claims, μ_N. Using the notation in the last section, we will have $Y = N$, $\mu_Y = \mu_N$, and $\sigma_Y = \sigma_N$. From Equations (2) or (4), full credibility is assigned when μ_N is sufficiently large. As noted before, we general do not have access to the value μ_N. In that case, we use the number of observed claims in place of μ_N.

Proposition 2.2 (Full credibility for claim frequency) Full credibility is assigned to estimate the claim frequency if the expected/observed number of claims is at least n_f, where

$$\boxed{n_f := \frac{y_p \sigma_N}{k}.}$$

Here, we can use $z_{(1+p)/2}$ to estimate y_p.

When N follows a Poisson distribution, then $\mu_N = \sigma_N^2$. The full credibility standard can be written as

$$\mu_N \geq \frac{y_p \sigma_N}{k} = \frac{y_p \sqrt{\mu_N}}{k} \Rightarrow \sqrt{\mu_N} \geq \frac{y_p}{k}.$$

By squaring both sides of the above inequality, we obtain the following:

Proposition 2.3 (Full credibility for Poisson claim frequency) Suppose N follows a Poisson distribution. Then, full credibility is assigned to estimate the claim frequency if the expected/observed number of claims is at least n_f, where

$$\boxed{n_f = \lambda := \left(\frac{y_p}{k}\right)^2.}$$

Example 2.1 An insurance company requires a coverage probability of 99% for the number of claims to be within 5% of the true expected claim frequency, how many claims in the recent period are required for full credibility? If the insurance company receives 2,890 claims this year from the risk group and the manual list of expected claim is 3,000, what is the updated expected number of claims next year? Assume the claim frequency distribution is Poisson and the normal approximation applies.

Solution:
From the statement, we know that $p = 0.99$ and $k = 0.05$. Using normal approximation, we have

$$\lambda = \left(\frac{z_{0.995}}{0.05}\right)^2 = \left(\frac{2.5758}{0.05}\right)^2 = 2,653.96.$$

Hence, 2,654 claims are required to assign full credibility to the data. Since $n = 2,890 > 2653.96$, full credibility is assigned. Hence, the updated estimate is 2,890, and the manual list is discarded.

Example 2.2 (2018 SOA STAM SAMPLE QUESTION 148) You are given that

(i) The number of claims follows a binomial distribution, $\text{Bin}(m, q)$.

(ii) The actual number of claims must be within 1% of the expected number of claims with probability 0.95.

(iii) The expected number of claims for full credibility is 34,574.

Calculate the expected number of claims needed for full credibility.

Solution:
From the statement, we have $p = 0.95$ and $k = 0.01$. Since $N \sim \text{Bin}(m, q)$, we have $n_f = \mu_N = mq = 34,574$ and $\sigma_N^2 = mq(1-q) = n_f(1-q)$. Hence, the full credibility requirement is given by

$$n_f = \frac{y_p \sigma_N}{k} = \frac{y_{0.95}\sqrt{n_f(1-q)}}{0.01} = \frac{z_{0.975}\sqrt{n_f(1-q)}}{0.01}.$$

Rearranging yields

$$1 - q = n_f\left(\frac{0.01}{z_{0.975}}\right)^2 = 34,574\left(\frac{0.01}{1.96}\right)^2 = 0.9 \Rightarrow q = 0.1.$$

2.2 Full Credibility for Claim Severity

We consider the credibility estimate on the expected claim size. Let $\{X_i\}_{i=1}^n$ be a sample of i.i.d. loss variables with common mean μ_X and variance σ_X^2. We shall use the sample mean $\bar{X}$ to estimate the expected claim size, μ_X. Hence,

$$Y = \bar{X} = \frac{\sum_{i=1}^n X_i}{n}.$$

In this case, $\mu_Y = \mu_{\bar{X}} = \mu_X$ and $\sigma_Y = \sigma_{\bar{X}} = \sigma_X/\sqrt{n}$. Using (2), the full credibility standard is given by

$$\mu_X \geq \frac{y_p \sigma_X/\sqrt{n}}{k}.$$

Rearranging the above yields the following:

Proposition 2.4 (Full credibility for claim severity) Full credibility is assigned to the experience $\bar{X}$ to estimate the claim severity if the number of claims n satisfies

$$\boxed{n \geq \left(\frac{y_p}{k}\right)^2 \left(\frac{\sigma_X}{\mu_X}\right)^2 = \lambda \mathrm{CV}_X^2.}$$

Intutively, when the number of observed claims exceeds $\lambda \mathrm{CV}_X^2$, the fluctuation from the estimate $\bar{X}$ is stabilized (since it has a small variance), and thus full credibility is assigned to the sample mean of the data.

Example 2.3 Individual loss amounts are independently and identically distributed with a Pareto distribution with shape parameter $\alpha = 6$ and scale parameter θ. Determine the number of claims required for the average severity of observed claims to be within 5% of the expected severity with probability $p = 0.95$.

Solution:

It is known that. for $X \sim \text{Pareto}(\alpha, \theta)$,

$$\mathbb{E}[X] = \frac{\theta}{\alpha - 1} \quad \text{and} \quad \mathrm{Var}[X] = \frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}.$$

Hence, with $\alpha = 6$,

$$\mathrm{CV}_X = \frac{\sqrt{\mathrm{Var}[X]}}{\mathbb{E}[X]} = \sqrt{\frac{\alpha}{\alpha - 2}} = \sqrt{1.5}.$$

The full credibility standard is thus

$$n = \lambda \mathrm{CV}_X^2 = \left(\frac{z_{0.975}}{0.05}\right)^2 \times 1.5 = \left(\frac{1.96}{0.05}\right)^2 = 2,304.8753.$$

Hence, 2,305 claims is needed for full credibility.

2.3 Full Credibility for Aggregate Loss

We consider the credibility estimate for the expected aggregate loss. This credibility estimate P is also known as the ***pure premium***, and the given prior estimate M is called the ***manual premium*** or ***manual rate***. In this case, the observed data is $Y = S$, where S is the aggregate loss variable with severity variable X and frequency variable N. Recall that

$$\mu_S = \mu_X \mu_N \quad \text{and} \quad \sigma_S^2 = \mu_N \sigma_X^2 + \mu_X^2 \sigma_N^2.$$

Substituting this into the full credibility standard (2), we obtain

$$\mu_N \geq \frac{y_p \sqrt{\mu_N \sigma_X^2 + \mu_X^2 \sigma_N^2}}{k \mu_X} = \frac{y_p \sqrt{\mu_N \mathrm{CV}_X^2 + \sigma_N^2}}{k} = \lambda \sqrt{\mu_N \mathrm{CV}_X^2 + \sigma_N^2}.$$

Square both sides, followed by dividing both sides by μ_N, we have the following full credibility standard:

Proposition 2.5 (Full credibility for aggregate loss) Full credibility is assigned to estimate the aggregate loss if the expected/observed number of claims is at least n_S, where

$$\boxed{n_S := \lambda \left(\mathrm{CV}_X^2 + \sigma_N \mathrm{CV}_N \right).}$$

If we assume N follows a Poisson distribution, then $\mu_N = \sigma_N^2$. In this case, $\sigma_N \mathrm{CV}_N = 1$. Hence, we have the following full credibility standard:

Proposition 2.6 (Full credibility for aggregate loss with Poisson frequency) Suppose N follows a Poisson distribution. Then, full credibility is assigned to estimate the aggregate loss if the expected/observed number of claims is at least n_S, where

$$\boxed{n_S := \lambda \left(1 + \mathrm{CV}_X^2 \right).}$$

Remark 2.7. Combining Propositions 2.3, 2.4, and 2.6, we see that when N follows a Poisson distribution,

$$\boxed{\begin{aligned} \text{Full Credibility Standard for } S = {} & \text{Full Credibility Standard for } N \\ & + \text{Full Credibility Standard for } X \end{aligned}}$$

Example 2.4 (2018 SOA STAM SAMPLE QUESTION 2) You are given that

(i) The number of claims has a Poisson distribution.
(ii) Claim sizes have a Pareto distribution with parameters $\theta = 0.5$ and $\alpha = 6$.
(iii) The number of claims and claim sizes are independent.
(iv) The observed pure premium should be within 2% of the expected pure premium 90% of the time.

Calculate the expected number of claims needed for full credibility.

Solution:
From Example 2.3, we know that

$$\mathrm{CV}_X^2 = \frac{\alpha}{\alpha - 2} = \frac{6}{6-2} = 1.5.$$

Using normal approximation, we have

$$\lambda = \left(\frac{z_{0.95}}{0.02}\right)^2 = \left(\frac{1.645}{0.02}\right)^2 = 6763.8617.$$

Since N follows a Poisson distribution, the full credibility standard is given by

$$\lambda(1 + \mathrm{CV}_X^2) = 6763.8617(1 + 1.5) = 16,912.6563.$$

Therefore, the expected number of claims needed for full credibility is 16,913.

Example 2.5 (**2018 SOA STAM SAMPLE QUESTION 39**) You are given the following information about a commercial auto liability book of business:

(i) Each insured's claim count has a Poisson distribution with mean Λ, where Λ has a gamma distribution with $\alpha = 1.5$ and $\theta = 0.2$.

(ii) Individual claim size amounts are independent and exponentially distributed with mean 5000.

(iii) The full credibility standard is for aggregate losses to be within 5% of the expected with probability 0.90.

Using limited fluctuated credibility, calculate the expected number of claims required for full credibility.

Solution:
Notice that $N \sim \mathrm{NB}(r = 1.5, \beta = 0.2)$. Hence, $\mu_N = (1.5)(0.2) = 0.3$ and $\mathrm{Var}[N] = (1.5)(0.2)(1+0.2) = 0.36$. On the other hand, for $X \sim \mathrm{Exp}(5000)$, we have $\mathbb{E}[X] = 5000$ and $\mathrm{Var}[X] = 5000^2$. With $p = 0.9$ and $k = 0.05$, the parameter λ is given by

$$\lambda = \left(\frac{z_{0.95}}{0.05}\right)^2 = \left(\frac{1.645}{0.05}\right)^2 = 32.9.$$

Hence, using Proposition 2.6, the full credibility standard is given by

$$\lambda\left(\mathrm{CV}_X^2 + \sigma_N \mathrm{CV}_N\right) = 32.9^2 \left(\frac{5000^2}{5000^2} + \frac{0.36}{0.3}\right) = 2,381.302.$$

Hence, the expected number of claims is 2,381.

3 Partial Credibility

When the full credibility standard (2) is not fulfilled, we need to determine the credibility factor $Z \in [0, 1)$. The credibility estimate will then depend on both the data Y and the prior estimate M.

Recall that the credibility estimate $P = ZY + (1 - Z)M$, where Y is the random variable that denotes the observation from the data. Let $k > 0$ be the range parameter and $p \in (0, 1)$ be the confidence interval. The credibility factor Z is determined such that the probability that ZY lies within $[Z\mu_Y - k\mu_Y, Z\mu_Y + k\mu_Y]$ is at least p:

$$\boxed{\mathbb{P}(|ZY - Z\mu_Y| \leq k\mu_Y) \geq p,}$$

which is equivalent to

$$\mathbb{P}\left(\left|\frac{Y - \mu_Y}{\sigma_Y}\right| \leq \frac{k\mu_Y}{Z\sigma_Y}\right) \geq p.$$

Hence, we have

$$Z = \frac{k\mu_Y}{y_p\sigma_Y}$$

Consider the estimation of claim frequency, i.e., $Y = N$. Assume further that N follows a Poisson distribution. Then $\mu_N = \mu_Y = \sigma_Y^2$, and the above condition reads as

$$Z = \frac{k\sqrt{\mu_N}}{y_p} = \sqrt{\frac{\mu_N}{\lambda}}, \tag{5}$$

where $\lambda = (y_p/k)^2$ is the full credibility standard for claim frequency (see Proposition 2.3). Therefore, if the expected number of claims $\mu_N < \lambda$, the credibility factor Z is given by the square root of the factor μ_N/λ; if $\mu_N \geq \lambda$, we shall take $Z = 1$, since Z cannot be greater than 1. In this case, full credibility is assigned. The formula (5) is also called the ***square root rule for partial credibility***:

Definition 3.1 (Square root rule for partial credibility) Let n be the true/expected number of claims in the experience, and n_0 be the true/expected number of claims for full credibility. Then, the credibility factor is determined by

$$\boxed{Z = \begin{cases} \sqrt{\dfrac{n}{n_0}}, & \text{if } n < n_0, \\ 1, & \text{if } n \geq n_0. \end{cases}}$$

Remark 3.1. The square root rule is not restricted to claim frequency estimation, it also applies to the estimation of claim severity, and aggregate loss. Depending on the quantity of interest, n_0 would be the full credibility standard for claim frequency, severity, or aggregate loss.

Example 3.1 You are given the following information for a group of insureds:

- Prior estimate of expected total losses 20,000,000.
- Observed total losses 25,000,000.
- Observed number of claims 10,000.
- Required number of claims for full credibility 17,500.

Using the methods of limited fluctuation credibility, determine the estimate for the group's expected total losses based upon the latest observation.

Solution:

From the information, we know that $M = 20,000,000$, $S = Y = 25,000,000$, $n_0 = 17,500$, and $n = 10,000$. The credibility factor Z is thus given by

$$Z = \sqrt{\frac{10000}{17500}} = 0.75593.$$

Hence, the credibility estimate is given by

$$P = ZS + (1-Z)M = (0.75593)(25,000,000) + (1-0.75593)(20,000,000) = 23,779,650.$$

Example 3.2 You are given the following:

- The number of claims follows a Poisson distribution.
- Claim sizes follow a gamma distribution with parameters $\alpha = 3$ and unknown θ.
- The number of claims and claim sizes are independent

The full credibility standard has been selected so that the actual claim costs will be within 8% of expected claim costs 95% of the time. If the expected number of claim is 2000, calculate the credibility factor.

Solution:

We first compute the full credibility standard n_0 for the aggregate loss. From Example 2.3, we know that

$$\mathrm{CV}_X^2 = \frac{\alpha}{\alpha - 2} = 3.$$

Hence, the full credibility standard is given by

$$n_0 = \lambda(1 + \mathrm{CV}_X^2) = \left(\frac{z_{0.95}}{0.08}\right)^2 (1+3) = 2401.$$

Using the square root rule, the credibility factor is given by

$$Z = \sqrt{\frac{2000}{2401}} = 0.9127.$$